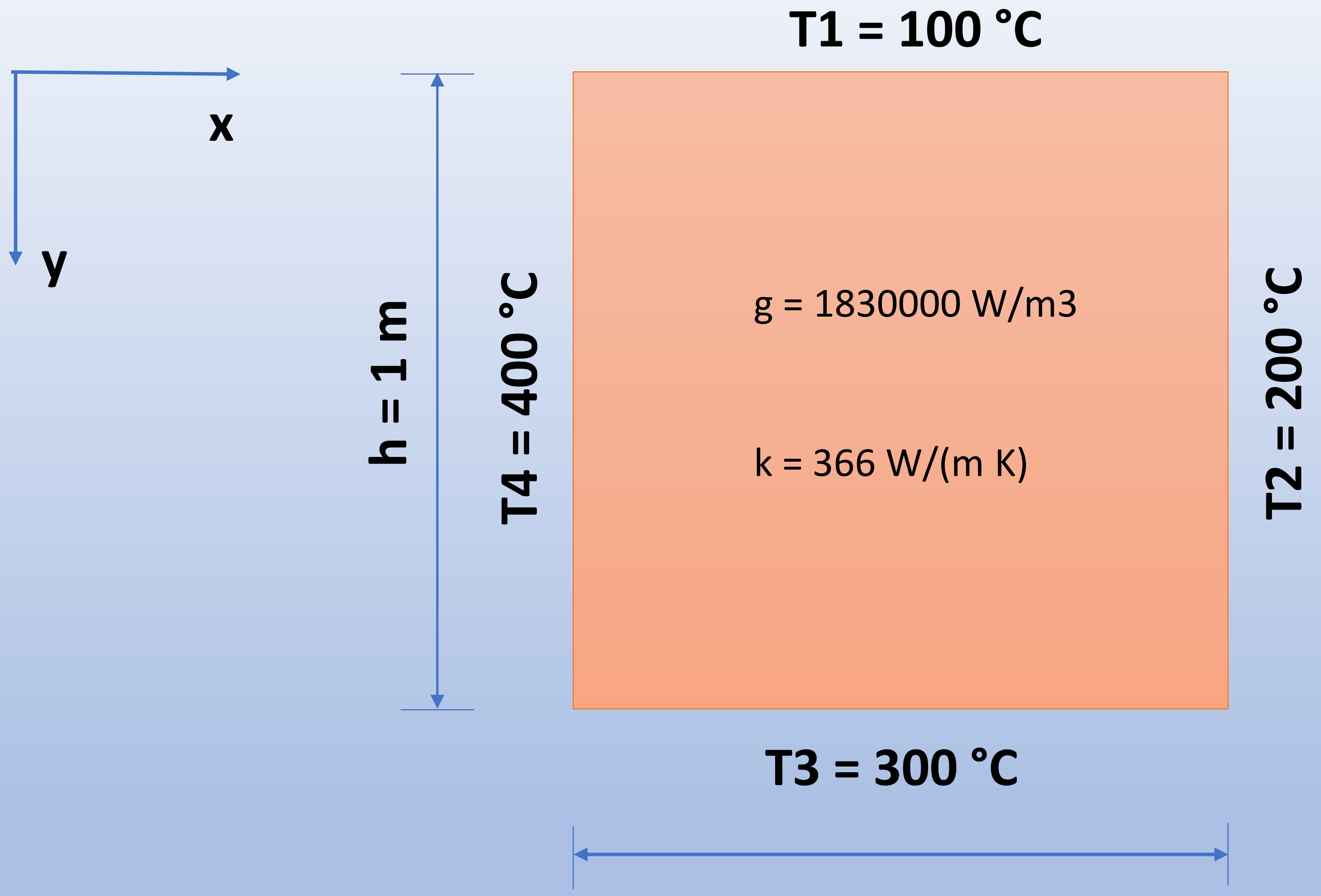
Solve 2D Steady State Heat Conduction Problem with heat generation in Cartesian Coordinates Using Finite Difference Method



Copper Plate

General Heat Conduction Equation in 3D Cartesian Coordinates

- $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ (1) ; T = T(x,y,z,t); x,y,z – spatial coordinates, t - time
- $\alpha - Thermal\ diffusivity, \frac{m2}{s}$
- $\alpha = k/(\rho * c)$
- k – thermal conductivity of the material , W/(m K)
- ρ - density of the material, kg/m3
- c – specific heat capacity of the material, J/(kg K)
- g – volumetric rate of internal heat generation, W/(m3)

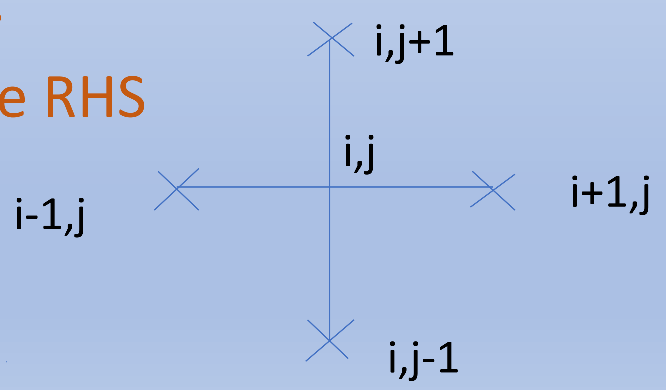
<u>Assumption of material thermal conductivity</u>

- kx, ky, kz does not vary along each of x, y or z axis (homogeneous)
- kx = ky = kz = k (isotropic)

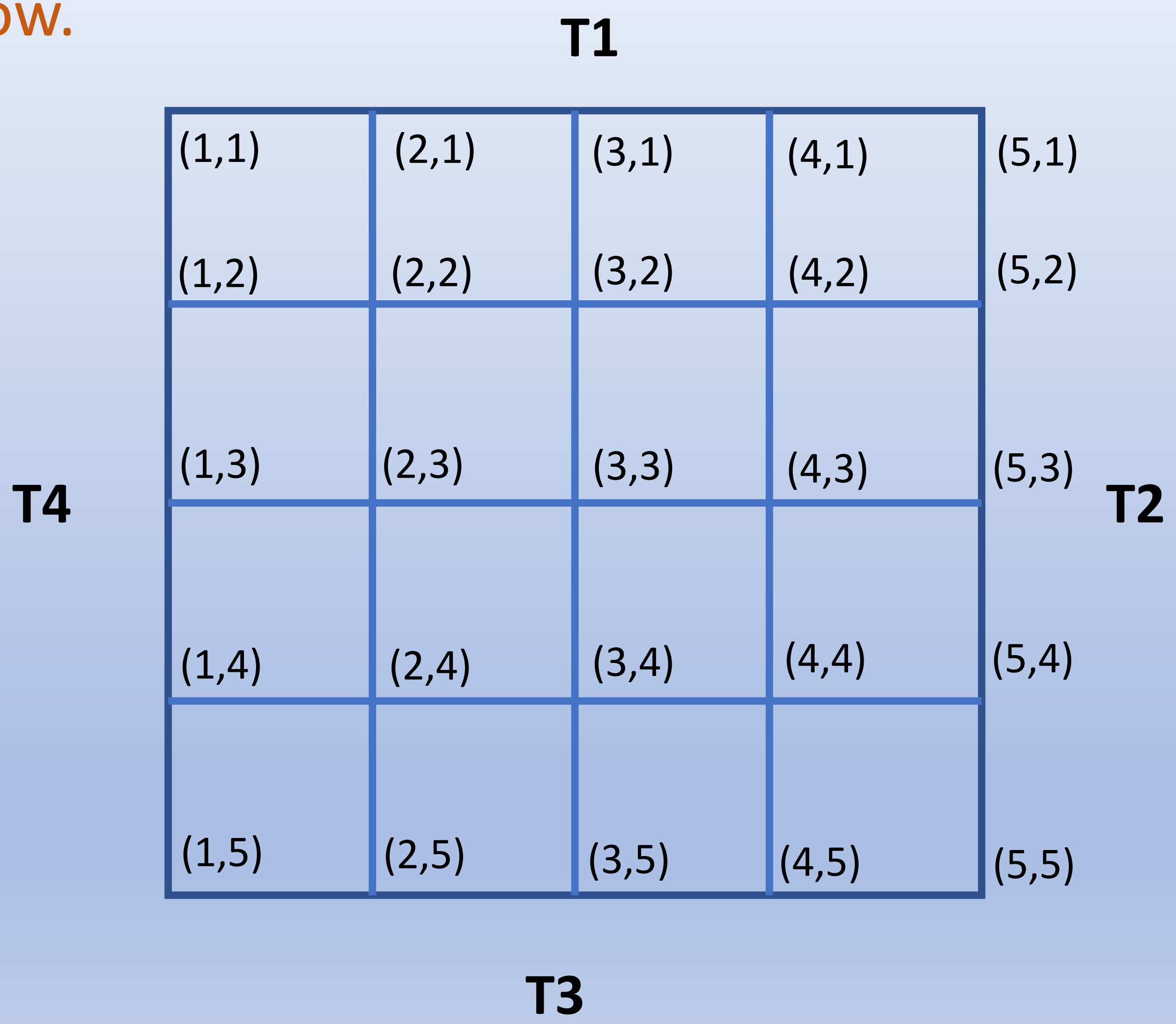
For 2D steady-state heat conduction with heat generation, equation (1) reduces to a simpler form

- We assume temperature does not vary significantly along z direction when compared with x and y directions.
- Temperature is independent of time (steady state)
- $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{g}{k} = 0$; T = T(x,y);
- BCs: T(x,y = 0) = T1; T(x=m,y) = T2; T(x,y=n) = T3; T(x=0,y) = T4 (2)
- m = length of the domain in x direction; n = length of the domain in y direction.
- The above equation is called Poisson equation in 2 dimensions.
- To obtain T, we need to solve the above PDE.
- We will utilize Finite Difference Method to solve the above PDE.
- To do so, we need to replace the partial derivatives with finite difference approximations
- We replace the space derivatives with second order centered difference approximations.

- $\left(\frac{T_{i-1,j} - 2 * T_{i,j} + T_{i+1,j}}{\Delta x^2}\right) + \left(\frac{T_{i,j-1} - 2 * T_{i,j} + T_{i,j+1}}{\Delta y^2}\right) + \frac{g}{k} = 0$
- For simplicity, let Δx = Δy
- $T_{i-1,j} + T_{i+1,j} - 4 * T_{i,j} + T_{i,j-1} + T_{i,j+1} = -\frac{g}{k} * \Delta x^2$; Eq (3)
- Equation (3) is the finite difference approximation of the original equation we were trying to solve.
- Here i represents the node location along the x direction and j represents the node location along the y direction.
- The finite difference stencil is on the RHS



- Now let us discretize the 2D domain into a 4 x 4 grid equally spaced, as shown below.



- We have 25 nodes in total
- Temperatures are fixed in the border nodes as shown
- Our interest is on the (9) interior nodes (2,2)……(4,4)
- Let us apply equation (3) on the (9) interior nodes
- $T_{1,2} + T_{3,2} - 4^* T_{2,2} + T_{2,1} + T_{2,3} = -\frac{g}{k} * \Delta x^2$; for (i= 2, j = 2)
- $T_{2,2} + T_{4,2} - 4^* T_{3,2} + T_{3,1} + T_{3,3} = -\frac{g}{k} * \Delta x^2$; for (i= 3, j = 2)
- $T_{3,2} + T_{5,2} - 4^* T_{4,2} + T_{4,1} + T_{4,3} = -\frac{g}{k} * \Delta x^2$; for (i= 4, j = 2)
- $T_{1,3} + T_{3,3} - 4^* T_{2,3} + T_{2,2} + T_{2,4} = -\frac{g}{k} * \Delta x^2$; for (i= 2, j = 3)
- $T_{2,3} + T_{4,3} - 4^* T_{3,3} + T_{3,2} + T_{3,4} = -\frac{g}{k} * \Delta x^2$; for (i= 3, j = 3)
- $T_{3,3} + T_{5,3} - 4^* T_{4,3} + T_{4,2} + T_{4,4} = -\frac{g}{k} * \Delta x^2$; for (i= 4, j = 3)
- $T_{1,4} + T_{3,4} - 4^* T_{2,4} + T_{2,3} + T_{2,5} = -\frac{g}{k} * \Delta x^2$; for (i= 2, j = 4)
- $T_{2,4} + T_{4,4} - 4^* T_{3,4} + T_{3,3} + T_{3,5} = -\frac{g}{k} * \Delta x^2$; for (i= 3, j = 4)
- $T_{3,4} + T_{5,4} - 4^* T_{4,4} + T_{4,3} + T_{4,5} = -\frac{g}{k} * \Delta x^2$; for (i= 4, j = 4)

- Boundary nodes have known temperatures and can be moved to the right side of the equations. The equations are re-arranged as below.

- $T_{3,2}$ - 4* $T_{2,2}$ + + $T_{2,3}$ = - $T_{1,2}$ - $T_{2,1}$ - $\frac{g}{k}$ * Δx^2;
- $T_{2,2}$ + $T_{4,2}$ - 4* $T_{3,2}$ + $T_{3,3}$ = - $T_{3,1}$ - $\frac{g}{k}$ * Δx^2;
- $T_{3,2}$ - 4* $T_{4,2}$ + $T_{4,3}$ = - $T_{4,1}$ - $T_{5,2}$ - $\frac{g}{k}$ * Δx^2;
- $T_{3,3}$ - 4* $T_{2,3}$ + $T_{2,2}$ + $T_{2,4}$ = - $T_{1,3}$ - $\frac{g}{k}$ * Δx^2;
- $T_{2,3}$ + $T_{4,3}$ - 4* $T_{3,3}$ + $T_{3,2}$ + $T_{3,4}$ = - $\frac{g}{k}$ * Δx^2;
- $T_{3,3}$ - 4* $T_{4,3}$ + $T_{4,2}$ + $T_{4,4}$ = $-$ $T_{5,3}$ - $\frac{g}{k}$ * Δx^2;
- $T_{3,4}$ - 4* $T_{2,4}$ + $T_{2,3}$ = - $T_{1,4}$ $-$ $T_{2,5}$ - $\frac{g}{k}$ * Δx^2;
- $T_{2,4}$ + $T_{4,4}$ - 4* $T_{3,4}$ + $T_{3,3}$ = - $T_{3,5}$ - $\frac{g}{k}$ * Δx^2;
- $T_{3,4}$- 4* $T_{4,4}$ + $T_{4,3}$ = - $T_{5,4}$ - $T_{4,5}$ - $\frac{g}{k}$ * Δx^2;

- The above equation can be arranged in a matrix form as below

- $$\begin{pmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} T_{2,2} \\ T_{3,2} \\ T_{4,2} \\ T_{2,3} \\ T_{3,3} \\ T_{4,3} \\ T_{2,4} \\ T_{3,4} \\ T_{4,4} \end{pmatrix} = - \begin{pmatrix} T_{1,2} + T_{2,1} \\ T_{3,1} \\ T_{4,1} + T_{5,2} \\ T_{1,3} \\ 0 \\ T_{5,3} \\ T_{1,4} + T_{2,5} \\ T_{3,5} \\ T_{5,4} + T_{4,5} \end{pmatrix} - \begin{pmatrix} \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \\ \frac{g}{k} * \Delta x^2 \end{pmatrix}$$
- The matrix shown above is a penta-diagonal matrix
- Iterative methods such as Gauss-Seidel, Successive Over Relaxation (SOR) method can be used to solve the above set of equations.
- Substituting the values of the boundary conditions, we get

$$
\begin{pmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{pmatrix}
\begin{pmatrix} T_{2,2} \\ T_{3,2} \\ T_{4,2} \\ T_{2,3} \\ T_{3,3} \\ T_{4,3} \\ T_{2,4} \\ T_{3,4} \\ T_{4,4} \end{pmatrix}
= -
\begin{pmatrix} 500 \\ 100 \\ 300 \\ 400 \\ 0 \\ 200 \\ 700 \\ 300 \\ 500 \end{pmatrix}
-
\begin{pmatrix} 312.5 \\ 312.5 \\ 312.5 \\ 312.5 \\ 312.5 \\ 312.5 \\ 312.5 \\ 312.5 \\ 312.5 \end{pmatrix}
= -
\begin{pmatrix} 812.5 \\ 412.5 \\ 612.5 \\ 712.5 \\ 312.5 \\ 512.5 \\ 1012.5 \\ 612.5 \\ 812.5 \end{pmatrix}
$$

- Graphical results are presented using MATLAB for this case.
- Using MATLAB or other software, we can develop codes for a general case where the number of grid spacings is, say m x n.
- We can then change the number of grid spacings as desired and obtain results accordingly

Summary

In this video,

- We presented a 2D Steady State heat conduction problem with heat generation in Cartesian Coordinates
- The temperature at each of the 4 sides of the square copper plate is fixed and Heat is generated within the domain.
- We solved the problem using Finite difference method and obtained the temperature profile
- We resolved the problem using smaller grid spacings and presented the results
- In future videos, we can explore more challenging problems.